

A Immirzi-like parameter for 3d quantum gravity

Valentin Bonzom and Etera R. Livine

Laboratoire de Physique, ENS Lyon, CNRS UMR 5672, 46 Allée d'Italie, 69364 Lyon Cedex 07, France

We study an Immirzi-like ambiguity in three-dimensional quantum gravity. It shares some features with the Immirzi parameter of four-dimensional loop quantum gravity: it does not affect the equations of motion, but modifies the Poisson brackets and the constraint algebra at the canonical level. We focus on the length operator and show how to define it through non-commuting fluxes. We compute its spectrum and show the effect of this Immirzi-like ambiguity. Finally, we extend these considerations to 4d gravity and show how the different topological modifications of the action affect the canonical structure of loop quantum gravity.

Introduction

Loop quantum gravity (LQG) presents a framework for a canonical quantization of general relativity (see e.g [1]). It defines a Hilbert space of quantum states of (space) geometry, spanned by the spin network states, and constraint operators implementing the invariance of the theory under space-time diffeomorphisms. It then derives discrete spectra for geometric quantities such as the areas and volumes. This whole framework is nevertheless affected by a quantization ambiguity, parametrized by the Immirzi parameter γ [2]. It is the parameter for a canonical transformation on the phase space, it does not have any effect on the classical equation of motion but it translates into a non-unitary transformation at the quantum level thus leading to non-equivalent quantization [2, 3]. It scales the spectra of geometric operators (the length unit gets a $\sqrt{\gamma}$ factor) but also affects the Hamiltonian constraint.

At the level of the classical action, Holst showed that the Immirzi parameter is introduced by adding a new term to the first order Palatini action for general relativity (GR) [4]. This new term is the square of the torsion up to the Nieh-Yan topological invariant. It does not change the equations of motion (as long as the metric is non-degenerate) which remain equivalent to the Einstein equations. Furthermore, in the McDowell-Mansouri reformulation of GR as a constrained BF theory for a $SO(4, 1)$ gauge group, γ amounts to introducing a quadratic potential for the Lagrange multiplier B [5]. Physically, it has also been shown that the Immirzi parameter is related to CP violations for fermion fields [6]. Finally, since the Immirzi parameter is due to an extra term in the classical action, it will affect the path integral even though it does not change the field equations. There has been recent proposals to take it into account in spin foam models for the LQG path integral [7].

In the present work, we study a similar ambiguity in 3d quantum gravity. From the point of view of the generalized Palatini action introduced by Holst [4], the Immirzi ambiguity appears because of the existence of two non-degenerate invariant bilinear forms on the Lorentz algebra $\mathfrak{so}(3, 1)$ (or $\mathfrak{so}(4)$ in the Riemannian case). The same idea is applied in three dimensions and again leads to an ambiguity in the action, as first noted by Witten in [8]. Gravity with a cosmological constant in three space-time dimensions is a topological BF field theory. We can then introduce an extra Chern-Simons-like term which does not modify the equations of motion for pure gravity. This term affects the path integral [8]. However, although 3d loop quantum gravity has been thoroughly studied [9, 11, 12], this Immirzi-like ambiguity has not been investigated in details at the canonical level as far as we know (it was nevertheless briefly mentioned in [9] as the θ -ambiguity). The natural questions is whether or not it leads to inequivalent loop quantizations like in the 4d theory and how does it affect the geometrical operators at the kinematical level.

We first review in section I some basic facts about 3d gravity reformulated as a Chern-Simons action for an extended $\mathfrak{so}(3, 1)$ -connection. We then introduce the Immirzi parameter for 3d gravity in section II. We perform the Hamiltonian analysis, carefully looking at the constraints algebra. The quantization is studied in section III, where we have to deal with non-commuting fluxes. We focus on the construction of the length operator. The Immirzi parameter alters its spectrum by a shift instead of the usual scaling derived in 4d loop quantum gravity. Moreover it gives rise to a new ambiguity: the curve whose length is measured has to be labeled with a $\mathfrak{su}(2)$ -representation. Finally, we extend these considerations to the 4d case in section IV and discuss the effect on the theory's canonical structure of the various topological terms that we can add to the action.

I. A QUICK REVIEW OF 3D GRAVITY

Let us consider a 3-manifold M and a principal G -bundle over M . We focus on $G = SU(2)$ for 3d Riemannian gravity. We call \mathfrak{g} the Lie algebra of G . Gravity is formulated in term of a triad field e and the spin connection ω . e is a \mathfrak{g} -valued one form and ω is a \mathfrak{g} -connection form whose curvature is denoted by $F[\omega]$. Internal indices are contracted

with the Killing metric δ_{ij} on \mathfrak{g} ($i, j = 1, 2, 3$). The action of 3d gravity with cosmological constant Λ reads:

$$S_{GR}(e, \omega) = \frac{1}{4\pi G} \int \left[2 e^i \wedge F_i[\omega] + \frac{\Lambda}{3} \epsilon_{ijk} e^i \wedge e^j \wedge e^k \right]. \quad (1)$$

In the following, we will systematically forget Newton's constant and set $4\pi G \equiv 1$. The equations of motion¹ impose a vanishing torsion, $d_\omega e^i = 0$, and a constant curvature given by the cosmological constant, $F^i + (\Lambda/2) \epsilon^i_{jk} e^j \wedge e^k = 0$. The action (1) is invariant under $SU(2)$ gauge transformations:

$$e \rightarrow geg^{-1}, \quad \omega \rightarrow g\omega g^{-1} + gdg^{-1},$$

with $g \in SU(2)$. It is also invariant under a translational symmetry:

$$\delta\omega^i = \Lambda \epsilon^i_{jk} e^j \chi^k \quad \text{and} \quad \delta e^i = d_\omega \chi^i. \quad (2)$$

This symmetry is also called 'topological' symmetry since it implies that the field e is pure gauge and is responsible for the lack of local degrees of freedom of the theory. Space-time diffeomorphisms can be generated as a combination of both types of transformations.

These two symmetries can be unified into a single gauge symmetry by enlarging the rotation group G to a larger gauge group \tilde{G} . This idea is the key of the reformulation of 3d gravity as a Chern-Simons theory for \tilde{G} [8]. The larger gauge group \tilde{G} is $SO(4)$, $ISO(3)$ or $SO(3, 1)$ depending on whether Λ is positive, zero or negative. The generators of the Lie algebra $\tilde{\mathfrak{g}}$ satisfy the commutation relations:

$$[J_i, J_j] = \epsilon_{ij}^k J_k, \quad [J_i, K_j] = \epsilon_{ij}^k K_k, \quad [K_i, K_j] = s \epsilon_{ij}^k J_k, \quad (3)$$

where $s = -1, 0, 1$ is the sign of the cosmological constant. We now define a connection for the enlarged gauge group, $A = \omega^i J_i + \sqrt{|\Lambda|} e^i K_i$. A rotation generated by an element $u = u^i J_i$ gives a $SU(2)$ gauge transformation, while boosts $v = v^i K_i$ give the translational symmetry. To build a Chern-Simons action, we choose a non-degenerate invariant bilinear form on the Lie algebra $\tilde{\mathfrak{g}}$:

$$\langle J_i, K_j \rangle = \delta_{ij} \quad \langle J_i, J_j \rangle = \langle K_i, K_j \rangle = 0. \quad (4)$$

Then, for a non-vanishing cosmologic constant $\Lambda \neq 0$, the action for 3d gravity can be written as a Chern-Simons theory:

$$S(A) = S_{GR}(\omega, e) = \frac{1}{\sqrt{|\Lambda|}} \int_M d^3x \epsilon^{\mu\nu\rho} (\langle A_\mu, \partial_\nu A_\rho \rangle + \frac{1}{3} \langle A_\mu, [A_\nu, A_\rho] \rangle) \quad (5)$$

The case of a vanishing cosmological constant is recovered by setting $s = 0$ and $\sqrt{|\Lambda|} = 1$ (or to any other arbitrary constant). The equations of motion simply say that the curvature R of the connection A vanishes. Since $R = (F^i + \frac{\Lambda}{2} [e, e]^i) J_i + (d_\omega e)^i K_i$, this is equivalent to the previous equations of motion with a vanishing torsion and a constant curvature.

II. THE IMMIRZI AMBIGUITY AT THE CLASSICAL LEVEL

A. Generalizing the action

When $\Lambda \neq 0$, there exists another invariant non-degenerate bilinear form on $\tilde{\mathfrak{g}}$. It is related to the one used above by the Hodge operator \star exchanging the rotations J_i with the boosts K_i : $(B, C) = \langle B, \star C \rangle$, with $(\star J)_{IJ} \equiv \epsilon^{IJ}_{KL} J^{KL}/2$. It is given explicitly by

$$(J_i, J_j) = \delta_{ij}, \quad (K_i, K_j) = s \delta_{ij} \quad \text{and} \quad (J_i, K_j) = 0. \quad (6)$$

¹ The covariant derivative is defined in term of the connection as $(d_\omega v)^i = dv^i + [\omega, v]^i = dv^i + \epsilon^i_{jk} \omega^j \wedge v^k$.

We define the associated Chern-Simons action :

$$\tilde{S}(A) = \frac{1}{\sqrt{|\Lambda|}} \int_M d^3x \epsilon^{\mu\nu\rho} \left((A_\mu, \partial_\nu A_\rho) + \frac{1}{3} (A_\mu, [A_\nu, A_\rho]) \right). \quad (7)$$

Let us write it in (e, ω) variables :

$$\begin{aligned} \tilde{S}(A) &= \frac{1}{\sqrt{|\Lambda|}} \int_M \omega^i \wedge d\omega_i + \frac{1}{3} \epsilon_{ijk} \omega^i \wedge \omega^j \wedge \omega^k + s|\Lambda| e^i \wedge d_\omega e_i \\ &= \frac{1}{\sqrt{|\Lambda|}} S_{CS}(\omega) + s\sqrt{|\Lambda|} \int_M e^i \wedge d_\omega e_i. \end{aligned} \quad (8)$$

It is straightforward to check that this action actually gives the same equations of motion as the gravity action (1) for $s \neq 0$. We now consider a linear combination of both actions, $S_\gamma(A) = S(A) + \gamma^{-1} \tilde{S}(A)$. The equations of motion are again equivalent to the Einstein equations :

$$\begin{cases} (F^i + s \frac{|\Lambda|}{2} \epsilon^i_{jk} e^j \wedge e^k) + s \frac{\sqrt{|\Lambda|}}{\gamma} d_\omega e^i = 0 \\ d_\omega e^i + \frac{1}{\gamma\sqrt{|\Lambda|}} (F^i + s \frac{|\Lambda|}{2} \epsilon^i_{jk} e^j \wedge e^k) = 0 \end{cases} \quad (9)$$

Providing that $\gamma^2 \neq s$, we get a one-parameter family of theories classically describing 3d gravity². We call γ the Immirzi parameter in analogy with the parameter entering the Holst action in 4d since they both appear through Hodge duality.

The case of a vanishing cosmological constant $\Lambda = 0$ works in a similar way as above. We can derive by formally setting $s = 0$ and $|\Lambda| = 1$. Then the second bilinear form (6) becomes degenerate and the new action $\tilde{S}(A)$ reduces to the Chern-Simons action $S_{CS}(\omega)$ for the spin connection.

The particular choices $\gamma^2 = s$ correspond to restricting the Chern-Simons connection to its self-dual or anti-self-dual component. Indeed, we can use the Hodge operator to split the connection into two: $A = A_+ + A_-$ with $\star A_\pm = \pm \sigma A_\pm$. We have $A_\pm^i = \omega^i \pm \sigma \sqrt{|\Lambda|} e^i$ where $\sigma^2 = s$, explicitly $\sigma = 1$ for $\Lambda > 0$ and $\sigma = i$ for $\Lambda < 0$. Then the full action also splits in two:

$$S_\gamma(A) = (\gamma^{-1} + s\sigma) S_{CS}(A_+) + (\gamma^{-1} - s\sigma) S_{CS}(A_-) \quad (10)$$

When γ is infinite, $\gamma^{-1} = 0$, $S_\gamma(A)$ is the original action $S(A)$ and the two parts of the action S_\pm have opposite coupling constants. This relation is responsible for the link between the Turaev-Viro invariant and the Reshetikhin-Turaev invariant [10]. For finite values of the Immirzi parameter, this no longer holds. In the following, we will always work at $\gamma^2 \neq s$.

B. Canonical analysis

To perform the Hamiltonian analysis, we assume that M is of the form $M = \Sigma \times \mathbb{R}$, where Σ is a two-dimensional smooth manifold of arbitrary topology. We choose arbitrary coordinates $x^a = (x^1, x^2)$ on the canonical surface Σ and complete it with a coordinate time x^0 on \mathbb{R} . Following this 2+1 splitting, we write the action as :

$$\begin{aligned} S_\gamma &= \int d^3x \left(2\epsilon^{ab} \delta_{ij} (e_b^i \partial_0 \omega_a^j + \frac{1}{2\gamma\sqrt{|\Lambda|}} \omega_b^i \partial_0 \omega_a^j + s \frac{\sqrt{|\Lambda|}}{2\gamma} e_b^i \partial_0 e_a^j) \right. \\ &\quad \left. + 2\epsilon^{ab} \delta_{ij} (\omega_0^j + s \frac{\sqrt{|\Lambda|}}{\gamma} e_0^j) D_a e_b^i + \epsilon^{ab} \delta_{ij} (e_0^i + \frac{1}{\gamma\sqrt{|\Lambda|}} \omega_0^i) (F_{ab}^j + \Lambda \epsilon^j_{kl} e_a^k e_b^l) \right) \end{aligned} \quad (11)$$

² There are all classically equivalent to pure gravity. The coupling to matter fields will depend on γ (see in appendix for more details). This is the same situation as in 3+1 dimensions where the Immirzi parameter affects the effective dynamics of fermions [6].

The kinematical terms of (11) involving time derivative ∂_0 determine the Poisson brackets. The symplectic structure explicitly depends on the parameter γ :

$$\{\omega_a^i(x), e_b^j(y)\} = \frac{1}{2} \frac{\gamma^2}{\gamma^2 - s} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x - y), \quad (12)$$

$$\{\omega_a^i(x), \omega_b^j(y)\} = \frac{\sqrt{|\Lambda|}}{2} \frac{s\gamma}{s - \gamma^2} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x - y), \quad (13)$$

$$\{e_a^i(x), e_b^j(y)\} = \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s - \gamma^2} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x - y) \quad (14)$$

The connection ω is still conjugate to the triad field e . But now both fields e and ω have become non-commutative. We recover the usual canonical structure, $\{\omega, e\} = \epsilon\delta$, $\{\omega, \omega\} = 0 = \{e, e\}$, in the limit $\gamma \rightarrow \infty$. On the other hand, to get the symplectic structure for $\Lambda = 0$, we need to set $|\Lambda| = 1$ and $s = 0$. Then the connection ω becomes commutative while the triad e remains non-commutative.

Notice that it is always possible to find two constants α and β such that the fields $\omega_a^i + \alpha e_a^i$ and $\omega_a^i + \beta e_a^i$ are both commutative and canonically conjugate to each other. However both fields are connections on Σ , and hence the loop quantization based on a connection and its conjugate vierbein can not be applied straightforwardly.

The two remaining terms in the action (11) give the hamiltonian. It is simply a linear combination of constraints imposed by the Lagrange multipliers e_0^i and ω_0^i :

$$\epsilon^{ab} D_a e_b^i = 0 \quad \text{and} \quad F_{ab}^i + s|\Lambda| \epsilon^i_{jk} e_a^j e_b^k = 0. \quad (15)$$

These constraints are the same as in usual gravity. They are first class and represent the Lie algebra $\tilde{\mathfrak{g}}$. We define the smeared constraints:

$$G(\lambda) = 2 \frac{\gamma^2 - s}{\gamma^2} \int_{\Sigma} d^2x \epsilon^{ab} \delta_{ij} \lambda^i(x) D_a e_b^j(x), \quad (16)$$

$$H(\lambda) = \frac{\gamma^2 - s}{\gamma^2 \sqrt{|\Lambda|}} \int_{\Sigma} d^2x \epsilon^{ab} \delta_{ij} \lambda^i(x) (F_{ab}^j(x) + s|\Lambda| \epsilon^i_{jk} e_a^j e_b^k(x)). \quad (17)$$

In the limit $\gamma \rightarrow \infty$, $G(\lambda)$ generates infinitesimal $\text{SO}(3)$ transformation corresponding to gauge transformations on A with parameter $\lambda^i J_i$ while $H(\lambda)$ generates the translations corresponding to gauge transformations on A with parameter $\lambda^i K_i$. Since the Immirzi parameter affects the Poisson bracket, it also modifies the commutation relation of the constraints $G(\lambda)$ and $H(\lambda)$. This results in a change of representation basis for the Lie algebra $\tilde{\mathfrak{g}}$. Indeed $G(\lambda)$ now induces the infinitesimal transformation defined by $\lambda^i (J_i - \gamma^{-1} K_i)$:

$$\{G(\lambda), \omega_a^i(x)\} = D_a \lambda^i(x) - \gamma^{-1} \epsilon^i_{jk} \sqrt{|\Lambda|} e_a^j(x) \lambda^k(x), \quad (18)$$

$$\{G(\lambda), \sqrt{|\Lambda|} e_a^i(x)\} = \epsilon^i_{jk} \sqrt{|\Lambda|} e_a^j(x) \lambda^k(x) - \gamma^{-1} D_a \lambda^i(x), \quad (19)$$

while $H(\kappa)$ is related to the gauge transformation $\kappa^i (K_i - s\gamma^{-1} J_i)$:

$$\{H(\kappa), \omega_a^i(x)\} = \epsilon^i_{jk} \sqrt{|\Lambda|} e_a^j(x) \kappa^k(x) - s\gamma^{-1} D_a \kappa^i(x), \quad (20)$$

$$\{H(\kappa), \sqrt{|\Lambda|} e_a^i(x)\} = D_a \kappa^i(x) - s\gamma^{-1} \epsilon^i_{jk} \sqrt{|\Lambda|} e_a^j(x) \kappa^k(x). \quad (21)$$

Then the constraints algebra naturally reflects the commutation relations of $\tilde{\mathfrak{g}}$ in the basis $(J_i - \gamma^{-1} K_i, K_i - s\gamma^{-1} J_i)$:

$$\{H(\kappa), H(\lambda)\} = s G([\kappa, \lambda]) - s\gamma^{-1} H([\kappa, \lambda]) \quad (22)$$

$$\{G(\lambda), H(\kappa)\} = H([\lambda, \kappa]) - s\gamma^{-1} G([\lambda, \kappa]) \quad (23)$$

$$\{G(\lambda), G(\kappa)\} = G([\lambda, \kappa]) - \gamma^{-1} H([\lambda, \kappa]). \quad (24)$$

It is clear that we can find two linear combinations of the constraints which act as the usual rotation and translation constraints. Therefore we expect that the physical Hilbert space of the quantum theory will not be changed. Nevertheless, observables and the action of the corresponding quantum operators will be modified by the Immirzi parameter. In the next section, we study the length operator in the context of a loop quantization.

C. Comparing with 3d Yang-Mills theory

In the limit case $\Lambda = 0$, the Poisson bracket simplifies and leads to the phase space of a system with non-commutative momenta:

$$\{\omega_a^i(x), \omega_b^j(y)\} = 0, \quad (25)$$

$$\{\omega_a^i(x), e_b^j(y)\} = \frac{1}{2} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x - y), \quad (26)$$

$$\{e_a^i(x), e_b^j(y)\} = -\frac{1}{2\gamma} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x - y). \quad (27)$$

This can be read directly from the action:

$$S_\gamma = 2 \int e^i \wedge F_i[\omega] + \frac{1}{\gamma} S_{CS}(\omega) = 2 \int \left(e^i + \frac{1}{2\gamma} \omega^i \right) \wedge dw_i + \left(e + \frac{1}{3\gamma} \omega \right) \wedge \omega \wedge \omega. \quad (28)$$

Notice that we can not re-absorb entirely the Chern-Simons term in a redefinition of the triad field e . The canonical variables are now ω_a^i and $\pi_i^a = \epsilon^{ab}(e_{bi} + \frac{1}{2\gamma} \omega_{bi})$. Let us insist on the fact that the momentum π does not transform as the triad e under $SU(2)$ -gauge transformation but as a connection. The translational symmetry is still generated by the constraint $H(\lambda) = 0$ while $SU(2)$ gauge transformed are now generated by the constraint $G(\lambda) + \frac{1}{\gamma} H(\lambda) = 0$. Notice that the local form of the latter is: $\epsilon^{ab} (D_a e_b^i + \frac{1}{2\gamma} F_{ab}^i) = D_a \pi_i^a + \frac{1}{2\gamma} \epsilon^{ab} \partial_a \omega_{bi} = 0$.

A simple example of a situation involving non-commutative momenta is the Landau problem of a particle in a magnetic field (see e.g. [13]). Because the Lagrangian couples the velocity of the particle to the vector potential, the canonical momentum is the velocity shifted by the vector potential. Thus the brackets between velocity components do not vanish and are proportional to the magnetic field. This well-known situation has a field-theoretical analog (see e.g. [14]) which is of importance for us : the 3d Yang-Mills theory with an additional Chern-Simons term. This theory, known as topologically massive gauge theory [14], yields a massive gauge boson. The interesting point for us is that the phase space is the same as in 3d gravity, the mass of the gauge boson being just the (inverse of the) Immirzi parameter.

Indeed, let us consider, without loss of generality, the following abelian action:

$$S_{TMYM}(A) = -\frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} + \frac{m}{2} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (29)$$

A_μ is a gauge field (of mass dimension 1/2) and m is the mass of the photon³. From the Hamiltonian point of view, when $m = 0$, the momentum conjugated to the spatial connection A_a ($a = 1, 2$) is as usual the electric field E_a . For $m \neq 0$, the momentum becomes $\Pi_a = E_a + \frac{m}{2} \epsilon_{ab} A_b$ and the Poisson brackets for the electric field is the same as those of the triad components with zero cosmological constant (27):

$$\{E_a(x), E_b(y)\} = m \epsilon_{ab} \delta^{(2)}(x - y), \quad (30)$$

with the identification $m = -1/(2\gamma)$. The form of the Hamiltonian is unchanged, $H = \frac{1}{2}(E^2 + B^2)$. Thus, the Chern-Simons term does not modify the classical energy but alters the relations between momenta and electric fields. In gravity, the Chern-Simons term preserves the gauge symmetries – in particular, the theory is still topological – but modifies the expressions of their generators.

The Gauss constraint, which generates the gauge transformations, reads $\partial_a E_a + m \epsilon_{ab} F_{ab} = \partial_a \Pi_a + m \epsilon_{ab} \partial_a A_b = 0$. In the usual quantization, Π (resp. π for 3d gravity) becomes a functional derivative with respect to the connection. But

³ First, one can check that the field equation for the vector $G_\mu = \epsilon_{\mu\nu\sigma} F^{\nu\sigma}$ is that of a free massive field:

$$(\partial^2 + m^2)G_\mu = 0.$$

At the quantum level, the Feynman propagator has a singularity at $p^2 = m^2$, showing that the photon has become massive. Explicitly, we get in the Landau gauge:

$$D_{\mu\nu} = \frac{1}{p^2 - m^2} \left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} + i \frac{m}{p^2} \epsilon_{\mu\nu\sigma} p^\sigma \right)$$

The representation theory of the (2+1)d-Poincaré algebra shows that the photon has one polarization per momentum. We refer to [14] for more details.

Π (resp. π) is a connection while a functional derivative should transform covariantly under a gauge transformation. Then the quantization of the Gauss constraint implies to work with states which are not exactly gauge invariant [15]. When dealing with a background independent theory such as 3d gravity, the loop quantization avoids those difficulties and provides a well-defined gauge invariant and diff-invariant state space.

Leaving aside the topological constraints, we now proceed to the quantization of 3d gravity with the Immirzi parameter at the kinematical level, by quantizing the algebra of loop variables.

III. QUANTIZATION OF LENGTHS WITH THE IMMIRZI PARAMETER

The quantization of 3d gravity has been extensively studied. In particular, it was one of the first example of the loop quantization programme [9, 12] and has since been completed (see e.g. [11], see also [10, 16] for quantum groups and/or Chern-Simons approaches). We first work with $\Lambda = 0$ and define modified flux observables which enable us to compute the length spectrum. This will generalize straightforwardly to the generic $\Lambda \neq 0$ case.

A. Quantizing the loop variables

As we have seen above, the Poisson brackets for $\Lambda = 0$ reduce to:

$$\begin{aligned} \{\omega_a^i(x), \omega_b^j(y)\} &= 0, \\ \{\omega_a^i(x), e_b^j(y)\} &= \frac{1}{2} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x-y), \\ \{e_a^i(x), e_b^j(y)\} &= -\frac{1}{2\gamma} \epsilon_{ab} \delta^{ij} \delta^{(2)}(x-y). \end{aligned} \quad (31)$$

Following the standard loop quantization, we would like to quantize the holonomy-flux algebra (see e.g. [17]). Since the brackets between ω 's components vanish, we build the kinematical Hilbert space as usual: we use gauge invariant wavefunctionals of the connection which depend only on the holonomies of ω along the edges of some finite graph embedded in the canonical surface Σ . A basis of this Hilbert space of quantum geometry states is provided by the spin network states. The difficulty now resides in the non-commutativity of the triad components. We need to quantize the fluxes of the triad field so that the quantum commutation relations reflect the non-vanishing Poisson brackets of e 's components. We could first try adding to the usual action of the flux a term which acts by multiplication. The latter usually acts on holonomies by Poisson brackets :

$$\widehat{\int_c e^i} \Big|_{\gamma=+\infty} = -i \left\{ \int_c e^i, \cdot \right\} \quad (32)$$

for a curve c . In order to faithfully represent the Poisson brackets (31), the right term to add would be a simple function of the connection proportional to $\gamma^{-1} \int_c \omega^i$. However, this choice is not consistent with the different transformation properties of the connection and triad fields.

Our solution is to consider other observables, such that the new term to add to the triad operators is simply the holonomy of ω along the curve c , whose action is well defined by multiplication on spin-network states. For a curve c going from x_0 to x_1 we call $U_{x_1}^{x_0} \in \text{SU}(2)$ (or $U(c)$) the holonomy of ω along c and we write $V_{x_0}^{x_1} \equiv U_{x_0}^{x_1}$ (or $V(c)$) for its inverse. We also assign a $\text{SU}(2)$ representation of spin j to the curve c and we denote the $\mathfrak{su}(2)$ generators $\tau_i = -i\sigma_i/2$ where $(\sigma_i)_{i=1,2,3}$ are the usual Pauli matrices. Following [18], we introduce the matrix $O(c)$ in the spin j representation, whose matrix elements are :

$$O(c)_{\beta}^{\alpha} = \int_c (V_{x_0}^y e_{|y} V_y^{x_1})_{\beta}^{\alpha}, \quad (33)$$

where the holonomies are taken in the spin j representation and $e_{|y} = e_a^i(y) dx^a \tau_i^{(j)}$. This observable $O(c)$ amounts to inserting the triad in the holonomy and integrate over the position of the insertion. Notice that when c is a closed curve, the quantity $\text{tr}_{(j)}(O(c))$ is actually the one-insertion loop variable as used in the original formulation of loop quantization [9]. It is $\text{SU}(2)$ gauge invariant and is also invariant under infinitesimal topological transformations, $\delta\omega = 0$ and $\delta e = d\omega\chi$, which do not move the basepoint x_0 of the loop c , $\chi(x_0) = 0$. We also introduce the adjoint matrix $O(c)^{\dagger}$:

$$O^{\dagger}(c) = - \int_c U_{x_1}^y e_{|y} U_y^{x_0}. \quad (34)$$

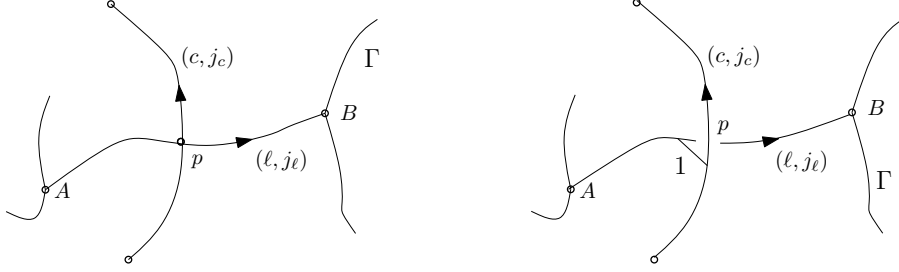


FIG. 1: The operator $\hat{O}(c)$ act on a spin-network graph Γ at the intersection point p . In the $\gamma \rightarrow \infty$ limit, it creates an intertwiner at p between the curves c and ℓ represented by a new (fictious) link labeled by the spin-1 representation, explicitly $-\frac{i}{2}(\tau^{(j_c)k})_b^a (\tau_k^{(j_\ell)})_\beta^\alpha$. For finite values of γ , we get a correction term to the intertwiner in γ^{-1} corresponding to the trivial intertwiner (with a spin-0 labeling the new link), explicitly $-\frac{i}{2}(\tau^{(j_c)k})_b^a (\tau_k^{(j_\ell)})_\beta^\alpha + \frac{1}{2\gamma}\delta_b^a\delta_\beta^\alpha$.

We first describe the action of $O(c)$ and $O^\dagger(c)$ at the quantum level when there is no Immirzi parameter $\gamma \rightarrow \infty$. They become operator-valued matrices, by replacing the triad field e by a functional derivative with respect to the connection. Since they consist in a single triad insertion, they act on holonomies locally. Assuming that the curve c crosses the graph of the spin-network graph Γ just once, at a point p belonging to the link ℓ , then the action of $\hat{O}(c)$ and $\hat{O}^\dagger(c)$ on the spin network state is given by their action on the matrix elements of the holonomy $U(\ell)_b^a$ along the link ℓ (taken in the spin j_ℓ representation), that is:

$$-i\{O(c)_\beta^\alpha, U(\ell)_b^a\} = -\frac{i}{2} \epsilon(c, \ell) \delta^{ik} \left(V_{x_0}^p \tau_i^{(j_c)} V_p^{x_1} \right)_\beta^\alpha \left(U_p^A \tau_k^{(j_\ell)} U_B^p \right)_b^a \quad (35)$$

$$-i\{O(c)^\dagger_\beta^\alpha, U(\ell)_b^a\} = \frac{i}{2} \epsilon(c, \ell) \delta^{ik} \left(U_{x_1}^p \tau_i^{(j_c)} U_p^{x_0} \right)_\beta^\alpha \left(U_p^A \tau_k^{(j_\ell)} U_B^p \right)_b^a \quad (36)$$

where A, B are respectively the start and target vertex of the link ℓ and the sign $\epsilon(c, \ell) = \pm$ gives the relative orientation of c and ℓ . Thus the curve c is added to the spin-network graph. The holonomies along c and ℓ are intertwined at the new vertex p by the following contraction of the $\mathfrak{su}(2)$ generators: $\sum_i \tau^{(j_c)i} \tau_i^{(j_\ell)}$. This defines an intertwiner $j_c \otimes j_\ell \rightarrow j_c \otimes j_\ell$ which can be decomposed as both representation j_c and j_ℓ coupling to the representation of spin 1, $((j_c \otimes \overline{j_c}) \rightarrow 1 \rightarrow (j_\ell \otimes \overline{j_\ell}))$. In the usual basis of $SU(2)$ representations with basis vectors $|j, m\rangle$, we have explicitly:

$$\begin{aligned} \sum_i (\tau_i)^{(j_1)}_{m_1 n_1} (\tau_i)^{(j_2)}_{m_2 n_2} &= -\frac{1}{4} \sqrt{j_1(j_1+1) - n_1(n_1+1)} \sqrt{j_2(j_2+1) - n_2(n_2-1)} \delta_{n_1, m_1-1} \delta_{n_2, m_2+1} \\ &\quad -\frac{1}{4} \sqrt{j_1(j_1+1) - n_1(n_1-1)} \sqrt{j_2(j_2+1) - n_2(n_2+1)} \delta_{n_1, m_1+1} \delta_{n_2, m_2-1} \\ &\quad -\frac{1}{4} m_1 m_2 \delta_{m_1, n_1} \delta_{m_2, n_2} \end{aligned} \quad (37)$$

Let us now move to finite values of γ . The Immirzi parameter deforms the quantization map:

$$\hat{O}(c)_\beta^\alpha = -i\{O(c)_\beta^\alpha, \cdot\} + \frac{1}{2\gamma} V(c)_\beta^\alpha \times \quad (38)$$

$$\hat{O}^\dagger(c)_\beta^\alpha = -i\{O^\dagger(c)_\beta^\alpha, \cdot\} + \frac{1}{2\gamma} U(c)_\beta^\alpha \times \quad (39)$$

This reproduces the algebra generated by the triad components. For instance, let us consider the bracket $\{O(c), O(c')\}$, with $\{p\} = c \cap c'$ being the only intersection point, c going from x_0 to x_1 , and c' from y_0 to y_1 . The term proportional to γ^{-1} is:

$$\int_c dx^a \int_{c'} dy^b (V_{x_0}^x \tau_i V_{x_1}^x)_\beta^\alpha (V_{y_0}^y \tau_j V_{y_1}^y)_\nu^\mu \{e_a^i(x), e_b^j(y)\} = -\frac{1}{2\gamma} \epsilon(c, c') (V_{x_0}^p \tau_i V_p^{x_1})_\beta^\alpha (V_{y_0}^p \tau^i V_p^{y_1})_\nu^\mu \quad (40)$$

We compare the commutator $[\hat{O}(c), \hat{O}(c')]$ computed from (38) to the Poisson brackets of $O(c)$ with the holonomy $V(c')$ shown in equation (35), and we easily check that the term proportional to γ^{-1} in $[\hat{O}(c), \hat{O}(c')]$ is indeed:

$-i/(2\gamma) (\{O(c), V(c')\} - \{O(c'), V(c)\})$. When c and c' do not intersect, $[\hat{O}(c), \hat{O}(c')]$ of course vanishes and there is no contribution from the correction terms in γ^{-1} .

Finally, we can view the quantum operators $\hat{O}(c)$ and $\hat{O}^\dagger(c)$ as acting on spin network functionals the same way as in the infinite γ limit, but with a modified intertwiner (see fig.1) taking into account the corrective holonomy term in the quantization map, $-\frac{i}{2} \sum_k (\tau_k^{(j_c)})_b^a (\tau_k^{(j_\ell)})_\beta^\alpha + \frac{1}{2\gamma} \delta_b^a \delta_\beta^\alpha$.

B. Length Spectrum

We now turn to the spectrum of lengths. For infinite γ it is given by the square root of the Casimir operator evaluated on the representation labeling the considered link ℓ , that is $\sqrt{j_\ell(j_\ell + 1)}/2$. For finite γ , we introduce the following operator:

$$\hat{S}_j(c) = \text{tr}_{(j)}(\hat{O}(c) \hat{O}^\dagger(c)) = -\text{tr}_{(j)}\left(\int_c V_{x_0}^y e_{|y} V_y^{x_1} \int_c U_{x_1}^{y'} e_{|y'} U_{y'}^{x_0}\right), \quad (41)$$

for a curve c going from x_0 to x_1 and colored by the spin j . This operator is obviously $\text{SU}(2)$ gauge invariant and its action is diagonalized by spin-network states:

$$\hat{S}_j(c)|\Psi\rangle = \frac{1}{4}\left(C(j) j_\ell(j_\ell + 1) - \frac{d_j}{\gamma^2}\right) |\Psi\rangle, \quad (42)$$

for a spin-network state $|\Psi\rangle$ and the curve c crossing the link ℓ just once. The coefficient $C(j)$ is the normalisation constant of the Killing metric in the spin j representation: $\text{tr}_{(j)}(\tau_i \tau_k) = C(j) \delta_{ik}$ with $C(j) = -j(j+1)d_j/3$ and $d_j = 2j+1$.

Let us see how the classical quantity $S_j(c)$ is related to the length $L(c)$ of the curve c . Assume that c is small i.e. of order $\epsilon \rightarrow 0^+$ in coordinates. Then we have $\int_c V_{x_0}^y e_{|y} V_y^{x_1} \approx \epsilon V_{x_0}^q e_{|q} V_q^{x_1}$, where q is an arbitrary point on c . To lowest order in ϵ , the holonomies can be approximated by the identity. Hence, $S_j(c) \approx C(j) \delta_{ik} \int_c e^i \int_c e^k$. Consider now a generic curve c . We follow the standard procedure in loop quantum gravity and cut it into N small pieces (c_n). The classical length (for a smooth curve c) is simply:

$$L(c) = \lim_{N \rightarrow +\infty} \sum_{n=1}^N \sqrt{|\delta_{ik} \int_{c_n} e^i \int_{c_n} e^k|} = \lim_{N \rightarrow +\infty} \sum_{n=1}^N \sqrt{|C(j)^{-1} S_j(c_n)|} \quad (43)$$

Since the operator $C(j)^{-1} \hat{S}_j(c_n)$ only receives contributions from intersection points, the limit of (42) when c_n becomes very small is fairly simple: the eigenvalue is zero if there is no intersection and is otherwise given by the r.h.s. of (42). The sum in (43) therefore contains a finite number of terms at the quantum level, one for each intersection of c with the graph of the spin-network state. For a single intersection, we get the following spectrum:

$$\text{Sp}(\hat{L}_j(c)) = \left\{ \frac{1}{2} \sqrt{j_\ell(j_\ell + 1) + \frac{3}{\gamma^2 j(j+1)}} , j_\ell \in \mathbb{N}^*/2 \right\}. \quad (44)$$

The effect of the Immirzi parameter is a simple shift of the usual spectrum. Therefore variations of the length given by differences between eigenvalues remain unaffected by γ . The only physical effect is that we will always have a non-zero minimal length in $1/\gamma \sqrt{j(j+1)}$ as soon as we measure a distance.

Furthermore, we see that the Immirzi parameter introduces a new ambiguity: classically both the length and the observable $C(j)^{-1} S_j(c)$ are naturally independent of the spin j labeling the curve c , however this spin j does not drop out of the eigenvalues of $C(j)^{-1} \hat{S}_j(c)$. We thus obtain a set of quantum length operators $(\hat{L}_j(c))_j$ labeled by the choice of a $\text{SU}(2)$ representation. This phenomenon has obviously no classical equivalent and we did not find any physical arguments allowing to fix j . This might be a mathematical ambiguity due to our choice of regularization. Or it might be a physical effect and j may depend on the observer or on the matter/particles used to define the end points of the curve c . Finally, we point out that the length shift goes in $1/j$ and vanishes in the limit $j \rightarrow \infty$.

It is possible to carry out the same analysis with a non-zero cosmological constant. As previously mentioned, we can find α such that $\omega + \alpha e$, which is a $\mathfrak{su}(2)$ connection on Σ , has vanishing Poisson brackets between its components. We use this connection to build the spin-network functionals. Then the length operator is obtained using the operators $O(c)$ and $O^\dagger(c)$ as above. Keeping track of the coefficients coming from the modified Poisson brackets, the eigenvalues become: $\frac{1}{2} |\frac{\gamma^2}{\gamma^2 - s}|^{1/2} \sqrt{j_\ell(j_\ell + 1) + \frac{3}{|\Lambda| \gamma^2 j(j+1)}}$. The previous spectrum is thus scaled by $\sqrt{|\frac{\gamma^2}{\gamma^2 - s}|}$ while the cosmological constant only appears in the j -dependent length shift proportional to γ^{-2} . Note that increasing the absolute value of Λ makes the j -dependent shift vanish. This can be understood directly from the bracket (14): since $\{e, e\} \propto 1/\sqrt{|\Lambda|}$, we recover a commutative triad in the limit $\Lambda \rightarrow \infty$.

IV. EXTENSION TO 4D LOOP GRAVITY

The introduction of the Immirzi parameter is formally identical in our 3d setting with a cosmological constant than in 4d: we use Hodge duality in $\mathfrak{so}(4)$ (resp. $\mathfrak{so}(3,1)$) to define the second bilinear form and add a new term in the action which does not change the equations of motion. More precisely, the variables in Riemannian (resp. Lorentzian) 4d gravity are a $\mathfrak{so}(4)$ -connection (resp. $\mathfrak{so}(3,1)$) A^{IJ} ($I, J = 1..4$) and a tetrad one-form e^I . We define the bivector fields $E^{IJ} = \frac{1}{2}\epsilon^{IJ}_{KL} e^K \wedge e^L$. The existence of two bilinear invariant forms then enables us to build the 4d Holst action as we did for 3d gravity in (7):

$$S_H(e, A) = 2 \int_M E^{IJ} \wedge (F + \frac{1}{\gamma} \star F)^{IJ} \quad (45)$$

$$= \int_M \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} + \frac{2}{\gamma} \int_M e_I \wedge e_J \wedge F^{IJ}, \quad (46)$$

with F the curvature of the connection A . Notice that unlike in 3d (8) with non-zero Λ , the second term in (46) alone does not give general relativity. Otherwise, similarly to eq.(10) for the 3d theory, the Immirzi parameter controls the relative contributions from the self-dual and anti-self-dual halves of the action. Indeed, let us project the fields E and A on their (anti-)self-dual parts with the projectors $(\text{id} \pm \star)$. In particular, we have $F(A_{\pm}) = F_{\pm}(A)$. We get:

$$S_H(e, A) = (1 + \frac{1}{\gamma}) \int_M E_{+IJ} \wedge F_+^{IJ} + (1 - \frac{1}{\gamma}) \int_M E_{-IJ} \wedge F_-^{IJ}. \quad (47)$$

Working with the bivectors E^{IJ} instead of the tetrad requires us to introduce the simplicity constraints ensuring that E comes from a tetrad (see e.g. [19, 20]). Alternatively one can use the bivector field \tilde{E}^{IJ} equal to $\frac{1}{2}\epsilon^{IJ}_{KL} e^K \wedge e^L + \gamma^{-1} e^I \wedge e^J$ [21, 22]. The Immirzi parameter is then absorbed in the (simplicity) constraints turning the topological BF theory into general relativity:

$$S_H(\tilde{E}, A) = \int \tilde{E}^{IJ} \wedge F_{IJ} - \frac{1}{2} \phi_{IJKL} \tilde{E}^{IJ} \wedge \tilde{E}^{KL} + \mu H, \quad (48)$$

where ϕ and μ are Lagrange multipliers such that $\phi_{IJKL} = -\phi_{JIKL} = -\phi_{IJLK} = \phi_{KLIJ}$, and $H = a_1 \phi_{IJ}^{IJ} + a_2 \phi_{IJKL} \epsilon^{IJKL}$. The parameters a_1 and a_2 are directly related to the Immirzi parameter: $a_2/a_1 = \frac{1}{4}(\gamma + \gamma^{-1})$. Such constraints do not have any equivalent in 3d for the theory is topological.

In 4d, the parameter γ is understood to control the amplitude of the fluctuations of the torsion in the path integral. Indeed the Immirzi term is simply the square of the torsion up to boundary terms⁴: $\int e^I \wedge e^J \wedge F_{IJ} = \int T^I \wedge T_I$ where $T^I = d_A e^I$ is the torsion. The relations between the Immirzi parameter and torsion have been thoroughly studied through the coupling of the 4d theory to fermions [6]. As we explain in appendix, the fermion coupling through the Immirzi parameter is different in our 3d setting than in the standard 4d analysis.

In the canonical framework, the Immirzi parameter induces a modification of the symplectic structure with a non-commutative connection [24]. Indeed, the simplicity constraints are second class and lead to Dirac brackets for which the triad is commutative but for which the components of the connection do not commute with each other anymore⁵.

Before looking for effects on the phase space of four dimensional gravity similar to those induced by the 3d parameter γ , let us mention a direct correspondence between γ and some parameters of the 4d BF theory with a cosmological term. The partition function for the theory under investigation is formally:

$$Z_{\gamma_{3d}} = \int DA e^{iS_{\gamma_{3d}}(A)} = \int DA_+ e^{\frac{i}{2\sqrt{\Lambda_{3d}}}(1+\gamma_{3d}^{-1})S_{CS}(A_+)} \times \int DA_- e^{-\frac{i}{2\sqrt{\Lambda_{3d}}}(1-\gamma_{3d}^{-1})S_{CS}(A_-)} \quad (50)$$

⁴ The difference of these two terms actually define the Nieh-Yan topological class defined as $\int d_A e^I \wedge d_A e_I - e_I \wedge e_J \wedge F^{IJ}(A)$.

⁵ The usual brackets of Loop Quantum Gravity result from a change of connection and a partial gauge fixing (the temporal gauge $e^0 = 0$). The Lorentz symmetry reduces to a $SU(2)$ subgroup [25, 26]. The symplectic structure is then very simple and the Immirzi parameter scales the only one non-vanishing bracket:

$$\{E_i^a(x), A_b^{(\gamma)j}(y)\} = \gamma \delta_i^j \delta_b^a \delta^{(3)}(x - y), \quad (49)$$

where E_i^a is the $\mathfrak{su}(2)$ -valued triad field and $A_a^{(\gamma)i}$ the Ashtekar-Barbero $\mathfrak{su}(2)$ -connection. This straightforwardly implies a scaling of the spectra of the geometrical operators. The areas, given like the lengths in 3d by the Casimirs of $\mathfrak{su}(2)$, are scaled as γ and the volumes as $\gamma^{3/2}$ (see e.g. [1]).

where we have denoted the 3d Immirzi parameter γ_{3d} and the 3d cosmological constant Λ_{3d} , here taken to be positive, to distinguish them from their 4d equivalents. To relate this partition function to another in 4d, we think about Chern-Simons theory in a four dimensional context. Indeed, given a $SO(4)$ -bundle over a 4-manifold M , the integral of the second Chern form, that is the Pontryagin class, equals the Chern-Simons action on the boundary ∂M . However, these quantities correspond to using the invariant non-degenerate bilinear form (\cdot, \cdot) , equivalent to the usual trace. Using instead the second bilinear form over $\mathfrak{so}(4)$, $\langle \cdot, \cdot \rangle$, yields the Euler class, which turns out to be similarly related to the Chern-Simons action for $\langle \cdot, \cdot \rangle$. Indeed:

$$F_{IJ}(A) \wedge F^{IJ}(A) = d\Omega_{CS}^{(\cdot)}(A_{\partial M}) \quad \text{and} \quad F_{IJ}(A) \wedge (\star F)^{IJ}(A) = d\Omega_{CS}^{(\cdot)}(A_{\partial M}) \quad (51)$$

where $\Omega_{CS}^{(\cdot)}$ and $\Omega_{CS}^{(\cdot)}$ are respectively the Chern-Simons 3-forms appearing in the actions S , (5), and \tilde{S} , (7). The partition function (50) can thus be interpreted in a four dimensional context as living on the boundary of the manifold:

$$Z_{\gamma_{3d}} = \int DA_{\partial M} e^{iS_{\gamma_{3d}}(A_{\partial M})} = \int DA_+ e^{\frac{i}{2\sqrt{\Lambda_{3d}}}(1+\gamma_{3d}^{-1}) \int_M F_{+IJ} \wedge F_+^{IJ}} \times \int DA_- e^{-\frac{i}{2\sqrt{\Lambda_{3d}}}(1-\gamma_{3d}^{-1}) \int_M F_{-IJ} \wedge F_-^{IJ}} \quad (52)$$

The second key observation is that these classes, which are bundle invariants, can be obtained in a first order formulation from the 4d BF theory with a cosmological term. Indeed, consider the following action:

$$S_{\gamma,\Lambda}(E, A) = \int_M E_{IJ} \wedge \left(F + \frac{1}{\gamma} \star F\right)^{IJ} - \frac{\Lambda}{2} E_{IJ} \wedge (\star E)^{IJ} \quad (53)$$

Notice that restricting E to be of the form $E = \star(e \wedge e)$ for a tetrad field e reproduces the Holst action $S_H(e, A)$ supplemented with a cosmological constant Λ . The partition function for $S_{\gamma,\Lambda}$ can be easily worked out using its self-dual/anti-self-dual decomposition:

$$S_{\gamma,\Lambda}(B, A) = \int_M B_{+IJ} \wedge F_+^{IJ} - \frac{\Lambda}{2(1+\gamma^{-1})^2} B_{+IJ} \wedge B_+^{IJ} - \int_M B_{-IJ} \wedge F_-^{IJ} - \frac{\Lambda}{2(1-\gamma^{-1})^2} B_{-IJ} \wedge B_-^{IJ} \quad (54)$$

in which we have rescaled the field E : $B_{\pm} \equiv (\gamma^{-1} \pm 1)E_{\pm}$. Formally performing the Gaussian integral over the field B , we are lead to the following action, which is a superposition of the Pontryagin and Euler classes:

$$S_{\gamma,\Lambda}(A) = \frac{(1+\gamma^{-1})^2}{2\Lambda} \int_M F_{+IJ} \wedge F_+^{IJ} - \frac{(1-\gamma^{-1})^2}{2\Lambda} \int_M F_{-IJ} \wedge F_-^{IJ} \quad (55)$$

As in the 3d case, the coupling constants of the self-dual and anti-self-dual parts do not simply defer by a sign because of the presence of the Immirzi parameter. Moreover, the interpretation of the action $S_{\gamma_{3d}}$ in terms of 4d topological classes enables to identify the coupling constants between the 3d and 4d cases, by comparing the partition function for $S_{\gamma,\Lambda}$ with (52). This leads to: $\gamma_{3d} = \frac{1}{2}(\gamma + \gamma^{-1})$, and $\sqrt{\Lambda_{3d}} = \Lambda\gamma^2/(1+\gamma^2)$.

Another interesting correspondence gives simple relations between the parameters. Consider the 4d BF action, but instead of the term $E \wedge \star F$ previously used, use the Hodge duality to introduce a new term quadratic in the field E :

$$S_{\beta,\Lambda}(E, A) = \int_M E_{IJ} \wedge F^{IJ} - \frac{\Lambda}{2} E_{IJ} \wedge \left(\star E + \frac{1}{\beta} E\right)^{IJ} \quad (56)$$

Notice however that the added term vanishes when evaluating it on the specific configurations $E = \star(e \wedge e)$. After integration over the field E into the path integral, we are again lead to the Pontryagin and Euler classes, but the coupling constants for the self-dual and anti-self-dual parts are now different functions of the parameters:

$$S_{\beta,\Lambda}(A) = \frac{1}{2\Lambda(1+\beta^{-1})} \int_M F_{+IJ} \wedge F_+^{IJ} + \frac{1}{2\Lambda(\beta^{-1}-1)} \int_M F_{-IJ} \wedge F_-^{IJ} \quad (57)$$

Comparing (57) and (52) gives the relations: $\gamma_{3d} = -\beta$ and $\sqrt{\Lambda_{3d}} = \Lambda(1 - \frac{1}{\gamma^2})$.

The 3d symplectic struture which we studied here can be extended to 4d by adding these bundle invariants to the 4d gravity action. In particular, instead of only considering the torsion squared, we should also consider terms involving the curvature squared. To this purpose, let us introduce the Pontryagin class and the Euler class into the 4d BF action:

$$S(A, E) = \frac{1}{2} \int_M E^{IJ} \wedge F_{IJ} + \theta_1 g_{IJKL} F^{IJ} \wedge F^{KL}, \quad (58)$$

$$\text{with } g_{IJKL} = \frac{1}{2}(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK}) + \frac{\theta_2}{2\theta_1} \epsilon_{IJKL}. \quad (59)$$

g_{IJKL} defines a metric on the Lie algebra $\mathfrak{so}(4)$. The Pontryagin and Euler classes do not depend on the connection (because of the Bianchi identity) and thus do not modify the equations of motion neither for BF theory nor for gravity (the Pontryagin class is the θ -term of Yang-Mills theories). θ_1 and θ_2 control the relative contributions of the self-dual and anti-self-dual parts of the action :

$$S(A, E) = \int_M \frac{1}{2} E_+^{IJ} \wedge F_{+IJ} + (\theta_1 + \theta_2) F_+^{IJ} \wedge F_{+IJ} + \int_M \frac{1}{2} E_-^{IJ} \wedge F_{-IJ} + (\theta_1 - \theta_2) F_-^{IJ} \wedge F_{-IJ}. \quad (60)$$

The canonical analysis of (58) is straightforward (see [27] for a detailed analysis) and is very similar to the 3d case that we studied in the previous sections: the momentum conjugated to the connection is $\Pi_{IJ}^a = E_{IJ}^a + 2\theta_1 B_{IJ}^a$, with $E_{IJ}^a = \epsilon^{abc} E_{bcIJ}$ and $B_{IJ}^a = \epsilon^{abc} g_{IJKL} F_{bc}^{KL}$. This yields a non-commuting E triad field:

$$\left\{ E_{IJ}^a(x), E_{KL}^b(y) \right\} = 4\theta_1 \left\{ \epsilon^{abc} g_{IJ}^{MN} D_c^{(x)} \delta_{MN}^{KL} \delta^{(3)}(x-y) - ((IJ) \leftrightarrow (KL), a \leftrightarrow b, x \leftrightarrow y) \right\}, \quad (61)$$

with $\delta_{MN}^{KL} = \frac{1}{2}(\delta_M^K \delta_N^L - \delta_N^K \delta_M^L)$. In this equation, the covariant derivative D_c is taken to act on the upper indices of δ_{MN}^{KL} . This bracket is thus proportional to the covariant derivative while in 3d, it is proportional to the identity (27). This is related to the fact that the momenta are shifted by the connection in 3d and by the curvature in 4d. As far as the phase space is concerned, notice a particular duality in 4d: while the torsion squared term $T^I \wedge T_I$ takes part in the non-commutativity of the connection, curvature squared terms $g_{IJKL} F^{IJ} \wedge F^{KL}$ are responsible for the non-commutativity of the triad field.

We now turn to GR adding these topological terms to the Palatini action and proceeding to the usual canonical analysis (following [25, 28]). Before imposing the second-class constraints, the (unreduced) phase space is that of BF theory. The Hamiltonian is however different, made of the Gauss, diffeomorphism and scalar constraints. One can easily check that the constraint algebra is not modified by the addition of the topological terms: any smearing of the bracket (61) over $\Sigma \times \Sigma$ identically vanishes.

Following [25], the second-class constraints are solved by writing E_{IJ}^a as $E_{IJ}^a = \frac{1}{2}(n_I E_J^a - n_J E_I^a)$ with a timelike unit vector n^I . The standard LQG approach relies on gauge fixing the time-like direction n with the choice $n^I \equiv (1, 0, 0, 0)$, which makes E_{IJ}^a a pure boost. The canonical variables are then the triad $E_i^a = E_{0i}^a$ and the extrinsic curvature $K_a^i = A_a^{0i}$. One should also solve the boost components of the Gauss constraint, which states that the rotational components of the connection form the $SU(2)$ spin-connection compatible with the triad E_i^a , that is $A_a^{ij} = -\epsilon^{ij}_k \Gamma_a^k(E)$. However, in the presence of the topological terms, the canonical momenta Π_{IJ}^a acquire a non-zero rotational part, let's call it B_{ij}^a . These components become functions $B_{ij}^a(E)$ of the triad after gauge fixing. Thus the canonical momenta of K_a^i and E_i^a have to be extracted from the following kinetic terms of the action (we have set $\theta_2 = 0$ for simplicity) :

$$S_{\text{kin}, \theta_1} = \int d^4x \left(E_i^a + 4\theta_1 \epsilon^{abc} \nabla_b K_{ci} \right) \partial_t K_a^i + 2\theta_1 \epsilon^{abc} \left(R_{bci} + \epsilon_{ijk} K_b^j K_c^k \right) \partial_t \Gamma_a^i, \quad (62)$$

∇ and R being respectively the covariant derivative operator and the curvature of the spin-connection $\Gamma(E)$. However, this result takes far from the 3d situation studied here and from the usual context of LQG.

We can nevertheless notice that the situation gets much simpler, and indeed very close to the 3d case, when looking at the self-dual formulation of gravity. In this case, we set the couplings $\gamma = 1$ and $\theta_1 = \theta_2$. The theory is then formulated in terms of $SU(2)$ variables right from the start (without any gauge fixing). Moreover, there is no additional second-class constraints to the Hamiltonian, the momentum Π_{+i}^a being an arbitrary self-dual field. Thus we only need to consider the self-dual terms of the action (60). We use the following notation, for all self-dual fields, $X^i = X_+^{0i}$, dropping the indice $+$ to emphasize the fact that all references to the anti-self-dual sector disappear. The phase space is now parametrized by pairs of canonically conjugate variables consisting of the connection A_a^i and a triad shifted by the curvature of A , $\Pi_i^a = E_i^a + 2\theta \epsilon^{abc} F_{bc i}$, with $\theta = 2\theta_1$. The resulting brackets are the same as those of BF theory with topological terms, for the group $SU(2)$:

$$\{A_a^i(x), A_b^j(y)\} = 0 \quad (63)$$

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^{(3)}(x-y) \quad (64)$$

$$\{E_i^a(x), E_j^b(y)\} = 4\theta \left[\epsilon^{abc} D_c^{(x)} \delta_{ij} \delta^{(3)}(x-y) - (a \leftrightarrow b, x \leftrightarrow y, i \leftrightarrow j) \right] \quad (65)$$

with $D_c \delta_{ij} = \partial_c \delta_{ij} + \epsilon_{ikj} A_c^k$. Notice that this is exactly the situation described in [30], but only for an Immirzi parameter fixed to $\gamma = 1$ (or similarly $\gamma = -1$). As shown in [30], the usual flux variables of LQG are undefined for such a canonical structure. Nevertheless, given the similarity of the phase space with the 3d structures that we studied, we propose an alternative strategy which could be fruitful to solve this issue: quantize another algebra

considering one-insertion loops variables instead of flux variables, so that the additional term required to satisfy the new commutation relations at the quantum level is simply given by a holonomy.

For arbitrary values of the couplings $\gamma, \theta_1, \theta_2$, we have to deal once again with the non-zero rotational part of Π_{IJ}^a as above for eq.(62). Indeed, considering the action:

$$S_{\gamma, \theta_1, \theta_2}(A, e) = \frac{1}{4} S_H(e, A) + \theta_1 \int F^{IJ} \wedge \left(F_{IJ} + \frac{\theta_2}{2\theta_1} \epsilon_{IJKL} F^{KL} \right), \quad (66)$$

one finds the following kinetic terms for the triad and the extrinsic curvature:

$$S_{\text{kin}, \gamma, \theta_1, \theta_2} = \int d^4x \frac{1}{\gamma} E_i^a \partial_t (\Gamma_a^i - \gamma K_a^i) + 4\theta_2 \epsilon^{abc} \nabla_b K_{ci} \partial_t \left(\Gamma_a^i - \frac{\theta_1}{\theta_2} K_a^i \right) - 2\theta_1 \epsilon^{abc} \left(R_{bci} + \epsilon_{ijk} K_b^j K_c^k \right) \partial_t \left(\Gamma_a^i - \frac{\theta_2}{\theta_1} K_a^i \right). \quad (67)$$

The first term of the r.h.s. is the usual one for LQG, while the second and the third are respectively the boost and the rotational parts coming from the kinetic terms of the topological classes. There is a special case $\gamma = \theta_1/\theta_2$ for which we can formulate the canonical structure can be formulated in term of the connection variables $\Gamma^\pm \equiv \Gamma - \gamma^{\pm 1} K$ as in Holst's analysis [4]:

$$S_{\text{kin}, \gamma = \frac{\theta_1}{\theta_2}} = \int d^4x \frac{1}{\gamma} \left(E_i^a + 4\theta_1 \epsilon^{abc} \nabla_b K_{ci} \right) \partial_t (\Gamma_a^i - \gamma K_a^i) - 2\theta_1 \epsilon^{abc} \left(R_{bci} + \epsilon_{ijk} K_b^j K_c^k \right) \partial_t \left(\Gamma_a^i - \frac{1}{\gamma} K_a^i \right). \quad (68)$$

As we see, the second kinetic term can be absorbed in a simple shift of the triad variable. Finally, we point out that the cases $\gamma = \pm 1$ are the only choices that make it possible to completely re-absorb the rotational components of Π_{IJ}^a in the momenta conjugate to $\Gamma_a^i - \gamma K_a^i$.

Conclusion

A Immirzi parameter for three-dimensional gravity can be formally introduced the same way as it appears in the four-dimensional Holst action for general relativity. However the Poisson brackets become more intricate in the 3d case and one has then to deal with a non-commuting triad field. The length spectrum can nevertheless be derived using modified flux operators. The contribution of the Immirzi parameter is not a scaling of the geometric spectra like in 4d, but a simple constant shift of the eigenvalues. The drawback of our approach is that we obtain a one-parameter family of length operators labeled by a $SU(2)$ representation which are all equivalent classically but not at the quantum level. This ambiguity is not yet fully understood and deserves further investigation. We can nevertheless compare with the regularization ambiguity for the quantization Hamiltonian constraint in 3+1d loop quantum gravity (see e.g. [17]).

In the final section, we compare our 3d setting and the standard 4d Immirzi parameter. Although they both come from the fact that there exists two bilinear forms on the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, they turn out to be rather different effects on the phase space. While the 4d Immirzi parameter relates to a torsion squared term in the action, our 3d Immirzi parameter is as expected better compared to the θ -parameters for the topological Pontryagin and Euler classes given as squares of the curvature. This should be related to recent investigation on the effect of a θ parameter in canonical loop gravity [30].

To conclude, we believe that the effects of this 3d Immirzi parameter on the observables of 3d BF theory and their state sum representation should be investigated in more details. This would establish further links between knot invariants and loop quantum gravity observables (in 3d). Furthermore, the moot point is whether or not it is possible to write a spin foam model of the Ponzano-Regge type (as a state sum) for 3d gravity with $\gamma \neq 0$. Indeed the levels of the (anti-)self-dual Chern-Simons theories are now different and the overall path integral is not given by the Turaev-Viro model anymore. Understanding how to deform the Turaev-Viro ansatz to accomodate such an extra term in 3d would provide a state sum representation of $SU(2)$ Chern-Simons theory but would also help understanding how to deal with similar deformations in 4d gravity.

Acknowledgements

We would like to thank Karim Noui for many discussions on 3d quantum gravity.

APPENDIX A: COUPLING TO FERMIONS

We have seen that the parameter γ labels a family of classically equivalent theories describing pure gravity. However, this is not true anymore in the presence of fermions because they are a source of torsion. In 4d like in 3d, the metric formulation uses the Levi-Civita connection which is torsion-free. In the Palatini first order framework with independent connection and vierbein fields, the vanishing torsion is implemented by the equation $d_\omega e = 0$. In 4d, fermions introduce torsion in the theory and the new equation of motion is of the form “ $d_\omega e = \text{fermionic current}$ ” [6] which explicitly involves the Immirzi parameter γ . Then it appears in the effective action for fermions as a coupling constant for a 4-fermions interaction (Einstein-Cartan term):

$$S_{\text{int}}(e, \psi) = -\frac{3}{2}\pi G \frac{\gamma^2}{\gamma^2 + 1} \int_M d^4x \sqrt{g} (\bar{\psi} \gamma_5 \gamma_I \psi) (\bar{\psi} \gamma_5 \gamma^I \psi). \quad (\text{A1})$$

In 3d, the parameter γ that we introduced is still related to torsion, although it plays a rather different role. Consider the following action for gravity at $\Lambda = 0$, restoring the Newton constant G :

$$S_\gamma(e, \omega) = \frac{2}{G} \int_M e^i \wedge F_i[\omega] + \frac{1}{\gamma} \int_M \omega^i \wedge d\omega_i + \frac{1}{3} \epsilon_{ijk} \omega^i \wedge \omega^j \wedge \omega^k. \quad (\text{A2})$$

Fermions transform under the fundamental representation of $\text{SU}(2)$ (there is neither Weyl spinors, nor axial and vector current in 3d). They are coupled to gravity through the following minimal coupling interaction:

$$S_F(e, \omega, \psi, \psi^\dagger) = \frac{i}{2} \int d^3x (\det e) (\psi^\dagger \sigma^i e_i^\mu D_\mu \psi - (D_\mu \psi)^\dagger \sigma^i e_i^\mu \psi) \quad (\text{A3})$$

where $\det e$ is the determinant of the triad and $D = d_\omega$, or explicitly $D_\mu = \partial_\mu - (i/2) \omega_\mu^i \sigma_i$, is the covariant derivation for ω . For $\gamma \rightarrow \infty$, the equation of motion for the torsion is:

$$\epsilon^{\mu\nu\lambda} D_\nu e_\lambda^i = -\frac{G}{4} (\det e) e^{\mu i} \psi^\dagger \psi, \quad (\text{A4})$$

which shows explicitly that fermions are a source of torsion in the theory. This equation is solved for the connection by writing ω as the sum of a torsion-free part $\Gamma[e]$, determined by $de + [\Gamma[e], e] = 0$, and a part containing the torsion: $\omega^i = \Gamma[e]^i + C^i$. Then the action can be expressed in terms of the triad and the fermionic fields. Here we get:

$$C_\mu^i = -\frac{G}{8} e_\mu^i \psi^\dagger \psi, \quad (\text{A5})$$

with a torsion of order G , like for 4d gravity. Inserting the expression of $\omega[e, \psi, \psi^\dagger]$ into the action, we recover general relativity with fermions in the second-order formalism with an additional interaction term:

$$S(e, \psi, \psi^\dagger) = \frac{1}{G} \int d^3x \epsilon^{\mu\nu\lambda} \delta_{ij} e_\mu^i F_{\nu\lambda}^j [\Gamma[e]] + \frac{i}{2} \int d^3x (e) (\psi^\dagger \sigma^i e_i^\mu \nabla_\mu \psi - (\nabla_\mu \psi)^\dagger \sigma^i e_i^\mu \psi) - \frac{3}{32} G \int d^3x (e) (\psi^\dagger \psi)^2, \quad (\text{A6})$$

where $\nabla_\mu = \partial_\mu - (i/2) \Gamma_\mu^i [e] \sigma_i$.

For 4d gravity, the first-order and second-order formalisms are not equivalent. They differ from each other due to the 4-fermion interaction (A1) which is proportional to the Newton constant G . This term remains even in the with $\gamma \rightarrow \infty$ limit. The situation is similar in 3d: torsion is responsible for a 4-fermion coupling of order G . This quartic interaction term is simply the squared fermionic density.

The situation becomes more intricate for finite values of γ . First the torsion equation of motion acquires a curvature term proportional to $G\gamma^{-1}$:

$$\epsilon^{\mu\nu\lambda} \left(D_\nu e_\lambda^i + \frac{G}{2\gamma} F_{\nu\lambda}^i [\omega] \right) = -\frac{G}{4} \det e e^{\mu i} \psi^\dagger \psi. \quad (\text{A7})$$

However the curvature is also non-zero in presence of fermions independently of the parameter γ . Varying the action with respect to the triad, we indeed have:

$$\epsilon^{\mu\nu\lambda} F_{\nu\lambda}^i [\omega] = -\frac{iG}{2} \epsilon^{\mu\nu\lambda} \epsilon^i_{jk} e_\nu^j e_\lambda^k \left(\psi^\dagger \sigma^k D_\lambda \psi - (D_\lambda \psi)^\dagger \sigma^k \psi \right) \quad (\text{A8})$$

Inserting this expression of $F[\omega]$ into (A7), we extract the correction to the connection ω due to the torsion:

$$C_\mu^i = \frac{1}{1 - \frac{G^2}{4\gamma} \psi^\dagger \psi} \left(i \frac{G^2}{4\gamma} (\psi^\dagger \sigma^i \nabla_\mu \psi - (\nabla_\mu \psi)^\dagger \sigma^i \psi) - \frac{G}{8} e_\mu^i \psi^\dagger \psi \right). \quad (\text{A9})$$

Putting this equation back into the action, one obtains an action which is not polynomial into the fermionic fields anymore. In particular, the denominator of C_μ^i imposes a limit to the fermionic density: $4|\psi^\dagger \psi| \leq G^2 \gamma^{-1}$. This is an intriguing new role for this 3d Immirzi parameter. Assuming that G very small, this denominator can be expanded in powers of G^2 and leads to a polynomial expansion of the action. At order G^2 , the denominator of C_μ^i can actually be neglected, and the action becomes:

$$\begin{aligned} S_\gamma(e, \psi, \psi^\dagger) = & \frac{1}{G} \int d^3x \epsilon^{\mu\nu\lambda} \delta_{ij} e_\mu^i F_{\nu\lambda}^j [\Gamma[e]] + S_F(e, \Gamma[e], \psi, \psi^\dagger) + \frac{1}{\gamma} S_{CS}(\Gamma[e]) \\ & - \frac{3}{32} G \int d^3x (e) (\psi^\dagger \psi)^2 - \frac{1}{8\gamma} G \int d^3x \epsilon^{\mu\nu\lambda} e_\mu^i F_{\nu\lambda}^i [\Gamma[e]] \psi^\dagger \psi \\ & + \frac{i}{4\gamma^2} G^2 \int d^3x \epsilon^{\mu\nu\lambda} (\psi^\dagger \sigma_i \nabla_\mu \psi - (\nabla_\mu \psi)^\dagger \sigma_i \psi) F_{\nu\lambda}^i [\Gamma[e]] + \dots \end{aligned} \quad (\text{A10})$$

It is thus clear that the 3d Immirzi parameter γ generates classically non-equivalent theories for gravity coupled to matter. Moreover it provides us with a new comparison with the 4d Immirzi parameter, which turns out to be related to torsion in a different way. In particular, in contrast with the 4d case, the torsion does not vanish in 3d in the limit $\gamma \rightarrow 0$.

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